Dirac Equation ... troubles

Reinhold Brückner
(*1963, Konstanz) rbr2018@gmx.com
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There are two entities in the world that Man cannot comprehend: Women - and the Dirac Equation. And I am not so sure about women.

The Dirac equation can be written as

$$[\gamma_0 D_t + \gamma_k D_k + m\gamma_m]\Psi = [DO]\Psi = 0$$

DO: Dirac Operator
γ : matrices with the properties : $[\gamma_a , \gamma_b]_+ = 0$ for a ≠ b ; $\gamma_0\gamma_0 = -1$; $\gamma_a\gamma_a = +1$ for a ≠ 0

$$D_t = i\partial_t + qA_0 \quad ; \quad D_k = i\partial_k + qA_k \quad ; \quad \partial_k = \frac{\partial}{\partial x_k} \quad ; \quad \partial_t = \frac{\partial}{\partial t}$$

A: vector potential, q: electric charge
A particular representation is

$$\gamma_0 = i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \gamma_k = \sigma_k\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma_m = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The matrices have 2 x 2 blocks as elements.
The σ_k are the Pauli matrices, generators of the SU(2) group:

$$\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One can see that the Dirac matrices are really two SU(2) groups stacked one upon the other.
The property $\gamma_0\gamma_0 = -1$ means that γ_0 is antihermitean. This makes the DO a sum of hermitean and antihermitean operators. This is why one has to use an adjoint spinor $\bar{\Psi} = \Psi^+\gamma_0$ to form the Dirac Lagrangean (DL)

$$L = \frac{1}{2}\bar{\Psi}[\gamma_0 D_t + \gamma_k D_k + m\gamma_m]\Psi + h.c.$$

From the DL, one obtains the definition of momentum and electric charge and current density.
momentum

$$P_k = \frac{1}{2}\bar{\Psi}\gamma_0 D_k \Psi + h.c.$$

current

$$J_k = q\bar{\Psi}\gamma_k\Psi$$

Let us form the equation of motion for momentum and current.
momentum

$$\partial_t \vec{P} = \frac{-1}{2}\partial_k \bar{\Psi}\gamma_k \vec{D}\Psi + h.c. + q\bar{\Psi}\gamma_0\Psi\vec{E} + q\bar{\Psi}\vec{\gamma}\Psi x \vec{B}$$

The first term is a surface term representing the momentum flow. The second term is the electric force acting on the charge density and the third term is the Lorentz force. Absolutely compatible with experiment.

Now for the electric current, defined as

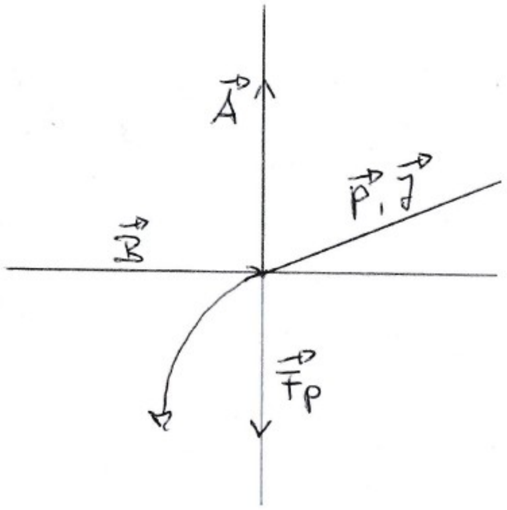
$$\vec{J} = q\,\bar{\Psi}\,\vec{\gamma}\,\Psi$$

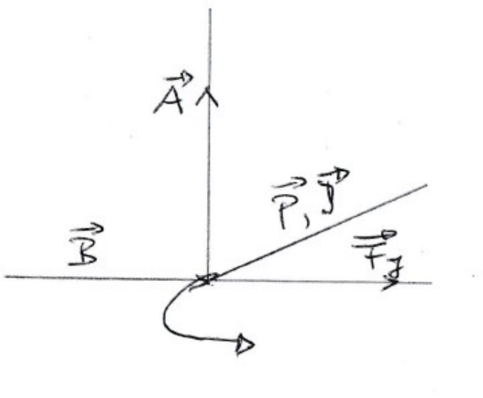
$$\partial_t \vec{J} = -q\,\nabla\bar{\Psi}\,\gamma_0\,\Psi - (i\,\nabla_{lf} + q\,\vec{A})\,xq\,\bar{\Psi}\,\gamma_0\,\vec{M}\,\Psi + h.c. + 2\mathrm{im}\,\Psi^{hc}\,\gamma_m\vec{\gamma}\,\Psi$$

($\Psi^{hc}=\Psi^{+}$ hc or h.c.= hermitean conjugate, x is the cross product, courtesy of a *** but free formula editor) ∇_{lf} means that the operator only acts on the left spinor $\bar{\Psi}$.

The RHS has three terms: The first one is some kind of a surface term.
The second one (plus h.c.) is an interaction of current momentum and external vector potential with its own magnetisation M.
The third term brings in the mass, without further comment.
This all very confusing, so let us calculate an example. We take a plane wave (ω,k), magnetised in its direction of motion. Such a wave has momentum k and current (k/ ω) which is the group velocity. This means that we can consider this the motion of a real particle.
With these conditions, the surface term and the gradient term (cross-product!) disappear. The mass term is zero as well.
We only care about the action of the external vector potential and compare it to its action on momentum.
If we choose a vector potential A in the x-direction to give a B-field in the y-direction and the momentum and current (and spin) moving (pointing) in the z-direction (see drawing), then
- the (Lorentz) force on momentum F_P points in the x-direction, parallel to A.
- the force on current F_J points in the y-direction, parallel to the B-field

That is *not exactly* what we expected.
And it is *not at all* what we observe.





We could, as is usually done, ignore that result and go on with our business. Who cares anyway. Problem is we observe the current rather than momentum in high-energy experiments. Momentum is an abstract quantity used to construct conservation laws but currents create electrodynamic fields and are the ones that interact with the apparatus.

We might now ask, how come we do not have this problem with the Schrödinger equation?
The reason is that the Schrödinger equation is not at all a low-energy limit to the Dirac equation. It is an approximation of the quadratic complex scalar field or an (incomplete) approximation of the Squared Dirac Operator.
The Squared Dirac Operator (SqDO) reads

$$[DO][DO]\Psi=[[-(i\partial_t+qA_0)(i\partial_t+qA_0)+(i\partial_k+qA_k)(i\partial_k+qA_k)+m^2]+qP_k E_k+q M_k B_k]\Psi=0$$

It consists of three parts:
- the complex scalar massive field
- the electric polarisation (P)
- the magnetisation (M)

For the Hydrogen atom, the scalar part gives the Balmer series with the Sommerfeld fine structure.
The electric polarisation gives intrinsic LS-interaction (Thomas precession).
The magnetisation gives Zeeman splitting of s-states in an external magnetic field.

The magnetisation is a matrix of the form

$M_k=\sigma_k\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ where the σ_k are the Pauli matrices and the matrix has 2x2 block elements

The electric polarisation is an anti-hermitean matrix

$P_k=i\gamma_0\gamma_k$ which can be brought in the form $P_k=\sigma_k\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$

The approximation to the Schrödinger equation concerns the quadratic scalar term of the covariant time derivative together with the mass term. Suppose a constant electric potential and time-Fourier-transform the term, one gets

$$(i\partial_t+qA_0)(i\partial_t+qA_0)-m^2\rightarrow(f+qA_0)(f+qA_0)-m^2=(f+qA_0+m)(f+qA_0-m)$$

Choose one frequency branch f = f'+m

$$(f+qA_0+m)(f+qA_0-m)=(f'+qA_0+2\mathrm{m})(f'+qA_0)\rightarrow(2\mathrm{m})(f'+qA_0)$$

then transform back to time space and get

$$[[-2\mathrm{m}(i\partial_t+qA_0)+(i\partial_k+qA_k)(i\partial_k+qA_k)]+qP_k E_k+q M_k B_k]\Psi=0$$

This is the Schrödinger-Dirac equation with electric and magnetic polarisation.
(Notice the anti-hermitean property of P)

While we are at it, let's look a little closer. We limited our approximation above to static electric potentials. If we have time-varying electric potentials, we can always write them as

$$A_0(\vec{r},t)=A_{0\mathrm{s}}(\vec{r})+A_{0\mathrm{d}}(\vec{r},t)$$

that is the sum of a static background and an oscillating potential.

We now perform a gauge transformation to eliminate the oscillating part from the electric potential. This will add a term to the vectorpotential

$$\vec{A}_d=\nabla\int^{t} dt'\,A_{0\mathrm{d}}(t')$$

Now only do we perform our approximation. This starts the treatment of oscillating fields, whether inductive or capacitive, at the same, mixed linear and quadratic level. It is little known that all energy and momentum transfer processes are quadratic at a minimum. These processes determine thermodynamics.

Back to relativity. We might ask whether we can use the SqDO as the starting point to cook up a (quadratic) Lagrangean from which we can define momentum and currents such as to avoid the kinematic contradiction of the Dirac Lagrangean. The anti-hermitean nature of the electric polarisation obliges us to use an adjoint spinor $\bar{\Psi} = \Psi^{+}\gamma_{\mathrm{m}}$. which necessitates 4-component spinors. That's not bad for a term that has never been discovered. (This also applies to the Schrödinger-Dirac equation)

In order to retain the advantage of the first-order Dirac equation, we can restrict the space of solutions of the SqDO to solutions of the DO.

With all this, the current of the SqDO has two parts: A magnetisation current $q\nabla x\vec{M}$ which we do not need to consider (it's a surface term) and, up to a coefficent, the momentum of the Dirac Operator. Its equation of motion, which we saw above, brings in the original DO-current. Hence, we push the problem back one layer but we do not eliminate it.

The low-energy approximation clearly shows that approximating an equation completely changes the whole mathematical and physical picture and is actually not an approximation anymore at all. Ironically, one then gets the correct physical behaviour.

So, we only have a problem at high energies.

And at high energies, the Dirac equation gives half good results (energies), half nonsense (kinematics). It will need *some* serious fixing.

Meaning that we do not have and never have had any, fully valid, relativistic wave equation for any matter field.